a-theorem in 6d

tensor branch of 6d SCFT's contains

- 4-derivative term:

$$b\,\frac{(\partial\psi)^4}{\psi^3} \subset \mathcal{L}_{\text{tensor}}$$

and

- 6-derivative term:

$$\Delta a\,\frac{(\partial\psi)^6}{\psi^6} \subset \mathcal{L}_{\text{tensor}}\,, \qquad \Delta a = a_{og} - (a_h + 1)$$

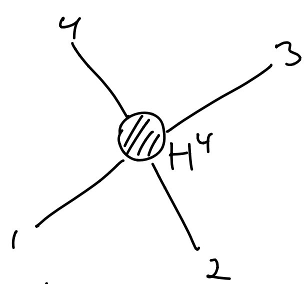
required by conformal symmetry

These can be obtained from scattering amplitudes:

- at 4 derivatives we have the 4-point, 4-derivative "supervertex" $\delta^8(Q)$:

$$\delta^8(\underset{\substack{\uparrow \\ \text{8 supercharges}}}{Q})$$

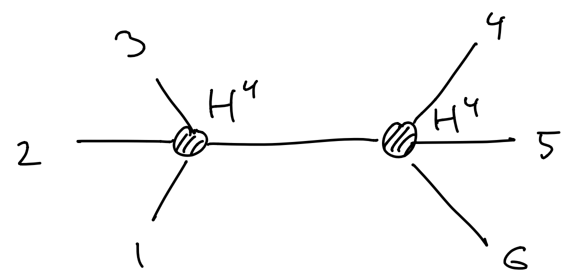
$$\delta^8(Q) = H^4 + (\partial\Phi)^4 + \cdots$$



$$\longrightarrow \; f_4(\Phi^I)\,\delta^8(Q)$$

n-point supervertex: $(n-4)$ sym. traceless tensor of $SO(5)_R$

- at 6 derivatives :

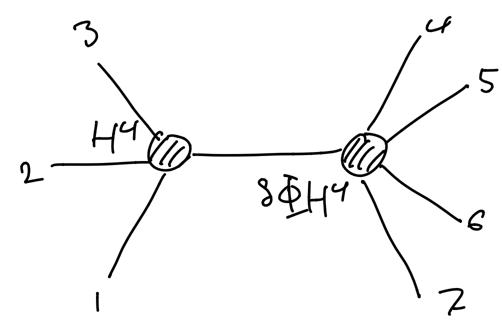


admits "factorization" through pair of 4-point supervertices $\delta^8(Q) \to f_4^2$

$\to$ coefficients of proportionality between f_6 and f_4^2 are fixed by supersymmetry

$\to$ $\Delta a \sim b^2$ (recall : $f_4(\Phi^I) = \frac{b}{|\Phi|^3}$)

- at 7 derivatives :



$\delta \Phi H^4$ arises by expanding $f_4(\Phi^I) H^4 = \dots$

$\to$ WZ - term (7 scalars, 6 derivatives)

5-point interaction arises from

$\partial_I f_4|_{\langle\Phi\rangle} \delta\Phi^I H^4$

$\to$ WZ- term $\sim \left(f_4|_{\langle\Phi\rangle}\right)\left(\partial_I f_4|_{\langle\Phi\rangle}\right)$

$\to$ $\Delta k \sim b^2$

Comparison to 5d SYM:

Recall:

- $$\mathcal{L}^{(5)}_{\text{Coulomb}} = -\frac{1}{2g^2}\Omega_{ij}\left(f_i \wedge * f_j + \sum_{I=1}^{5} \partial_\mu \varphi_i^I \partial^\mu \varphi_j^I\right)$$
$$+ (\text{Fermions}) + \cdots$$
$$\Omega_{ij} = \mathrm{Tr}_{\mathfrak{g}}(h_i h_j)$$

- relation to 6d fields:
$$\Phi_i^I \longrightarrow \frac{1}{2\pi R}\varphi_i^I, \quad H_i \longrightarrow \frac{1}{2\pi R}\left(f_i \wedge dx^5 + *^{(5)} f_i\right)$$
$$-\frac{\pi R}{g^2}\Omega_{ij}\left(H_i \wedge * H_j + \sum_{I=1}^{5}\partial_\mu \Phi_i^I \partial^\mu \Phi_j^I\right)$$
$$+ (\text{Fermions}) \subset \mathcal{L}_{\text{tensor}}$$

Restrict to <u>single</u> Abelian vector multiplet arising from breaking $\mathfrak{g} \longrightarrow \mathfrak{h} \oplus \mathfrak{u}(1)$

$\longrightarrow$ vev $\langle \varphi^I \rangle$ break $SO(5)_R \longrightarrow SO(4)_R$

$$\varphi = \left(\sum_{I=1}^{5} \varphi^I \varphi^I\right)^{\frac{1}{2}}$$

$$\phi^I = t\,\varphi^I, \quad F = t f, \quad t \in \mathfrak{t}_{\mathfrak{g}}$$

where $t \in \mathfrak{t}_{\mathfrak{g}}$ is a Cartan generator whose commutant in $\mathfrak{g}$ is $\mathfrak{h} \oplus \mathfrak{u}(1)$

$$\to -\frac{1}{4g^2}\,\mathrm{Tr}_{\mathfrak{g}}(t^2)\left(f\wedge * f + \sum_{i=I}^{5}\partial_\mu \varphi^I \partial^\mu \varphi^I\right) + (\text{Fermions})$$

$$\subset \mathcal{L}^{(5)}_{\text{Coulomb}}$$

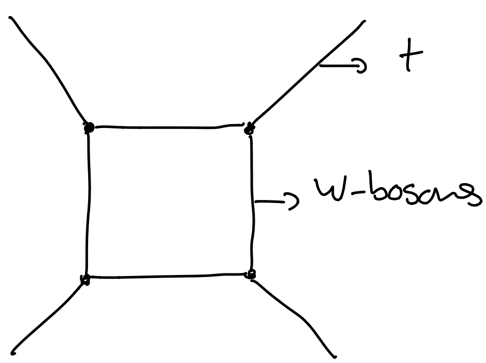
Higher derivative terms:

$$\left(\frac{b^{(5)}}{\varphi^3} + c^{(5)} g^6\right)(\partial\varphi)^4 \subset \mathcal{L}^{(5)}_{\text{Coulomb}}$$

$\uparrow$ not forbidden as in 6d as 5d theory is not conformal!

By comparing with 6d effective action $\mathcal{L}_{\text{tensor}}$, we see $c^{(5)} = 0$

$b^{(5)}$ arises from integrating out W-bosons as follows:



$$\sim (\alpha(t))^4 \int \frac{d^d p}{(p^2+m_W^2(\alpha))^4}$$

$$\sim (\alpha(t))^4 m_W^{d-8}(\alpha)$$

$$\sim |\alpha(t)|^{d-4} |\langle\varphi\rangle|^{d-8}$$

W-bosons are labeled by roots $\alpha \in \Delta_{\mathfrak{g}}$

that do <u>not</u> reside in the root system of $\mathfrak{h}$:

$\alpha \in \Delta_{\mathfrak{g}} / \Delta_{\mathfrak{h}}$, $U(1)$ charges $= \alpha(t)$

$$\longrightarrow b^{(5)} = \frac{1}{128\pi^2} \sum_{\alpha \Delta_{\mathfrak{g}}/\Delta_{\mathfrak{h}}} |\alpha(t)| \qquad (*)$$

Comparing with 6d action, we get

$$b = \left(\frac{g^2}{2\pi R \, \mathrm{Tr}_{\mathfrak{g}}(t^2)} \right)^{\frac{1}{2}} b^{(5)}$$

substituting $g^2 = 4\pi^2 R$ and $(*)$ gives:

$$b = \left(\frac{1}{8192\pi^3 \, \mathrm{Tr}_{\mathfrak{g}}(t^2)} \right)^{\frac{1}{2}} \sum_{\alpha \in \Delta_{\mathfrak{g}}/\Delta_{\mathfrak{h}}} |\alpha(t)|$$

<u>Computing $a_{\mathfrak{g}}$ and $k_{\mathfrak{g}}$:</u>

Upon breaking $\mathfrak{g} \longrightarrow \mathfrak{h} \oplus \mathfrak{u}(1)$ the anomaly difference between the UV theory $\mathcal{T}_{\mathfrak{g}}$ and the IR theory $\mathcal{T}_{\mathfrak{h}} \oplus \mathfrak{u}(1)_{AVM}$ is

$$\Delta a = a_{\mathfrak{g}} - (a_{\mathfrak{h}} + 1) = \frac{98304\pi^3}{7} b^2 ,$$

$$\Delta k = k_{\mathfrak{g}} - k_{\mathfrak{h}} = 6144\pi^3 b^2$$

Substituting 6 we find:

$$\Delta a = \frac{12}{7} X, \qquad \Delta K = \frac{3}{4} X,$$

$$X = \frac{1}{\mathrm{Tr}_{\mathfrak{g}}(f^2)} \left(\sum_{\alpha \in \Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{h}}} |\alpha(f)| \right)^2$$

One can show:

$$X = 16 \left(\langle \rho_{\mathfrak{g}}, \rho_{\mathfrak{g}} \rangle_{\mathfrak{g}} - \langle \rho_{\mathfrak{h}}, \rho_{\mathfrak{h}} \rangle_{\mathfrak{g}} \right)$$

where $\rho_{\mathfrak{g}} = \frac{1}{2} \sum_{\alpha \in \Delta^+_{\mathfrak{g}}} \alpha$, $\rho_{\mathfrak{h}} = \frac{1}{2} \sum_{\alpha \in \Delta^+_{\mathfrak{h}}} \alpha$

Furthermor, the following holds

$$\langle \rho_{\mathfrak{g}}, \rho_{\mathfrak{g}} \rangle_{\mathfrak{g}} = \frac{1}{12} h^\vee_{\mathfrak{g}} d_{\mathfrak{g}}$$

We have $\langle \cdot, \cdot \rangle_{\mathfrak{g}} = \underset{\uparrow}{N^{(i)}_{\mathfrak{h} \subset \mathfrak{g}}} \langle \cdot, \cdot \rangle_{\mathfrak{h}}$

normalization factor

$$\longrightarrow \Delta a = \frac{16}{7} \left(h^\vee_{\mathfrak{g}} d_{\mathfrak{g}} - \sum_i N^{(i)}_{\mathfrak{h} \subset \mathfrak{g}} h^\vee_{\mathfrak{h}_i} d_{\mathfrak{h}_i} \right)$$

for $\mathfrak{h} = \bigoplus_i \mathfrak{h}_i$

$$\Delta K = h^\vee_{\mathfrak{g}} d_{\mathfrak{g}} - \sum_i N^{(i)}_{\mathfrak{h} \subset \mathfrak{g}} h^\vee_{\mathfrak{h}_i} d_{\mathfrak{h}_i}$$

For $\mathfrak{g}$ of ADE type : $N^{(i)}_{\mathfrak{h} \subset \mathfrak{g}} = 1$

$\rightarrow a_{\mathfrak{g}} = \frac{16}{7} h^{\vee}_{\mathfrak{g}} d_{\mathfrak{g}} + 1 \,, \quad K_{\mathfrak{g}} = h^{\vee}_{\mathfrak{g}} d_{\mathfrak{g}} \,, \quad \mathfrak{g} \in \{A_n, D_n, E_n\}$

We need $\Delta K = K_{\mathfrak{g}} - K_{\mathfrak{h}} \in 6\mathbb{Z}$

$\rightarrow$ holds for ADE case as $h^{\vee}_{\mathfrak{g}} d_{\mathfrak{g}} \in 6\mathbb{Z}$ there!

<u>but</u> fails otherwise

<u>Example</u>: $\mathfrak{g} = \mathfrak{g}_2 \qquad \circ \equiv \bullet$

Higgsing leads to

$$N_{su(2)_\bullet \subset \mathfrak{g}_2} = \frac{1}{3} \,, \qquad N_{su(2)_\circ \subset \mathfrak{g}_2} = 1$$

$\rightarrow \Delta K_{su(2)_\bullet \subset \mathfrak{g}_2} = 4 \cdot 14 - \frac{1}{3} \cdot 2 \cdot 3 = 54 \,,$

$\Delta K_{su(2)_\circ \subset \mathfrak{g}_2} = 4 \cdot 14 - 2 \cdot 3 = 50$

not divisibly by 6!

$\rightarrow$ (2,0) theory $T_{\mathfrak{g}_2}$ is ruled out!